BIJECTIONS BETWEEN INDECOMPOSABLE SUMMANDS OF BASIC TILTING-TYPE MODULES

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Bonjour a tous / Good morning everyone

THANK you to the

- Organizers of NCRA VIII
- Participants from many countries

- What I will present is joint work with MELIS TEKIN AKCIN.
- What I will show you:

- quivers = oriented graphs
- Auslander-Reiten quivers of algebras = quivers such that.....

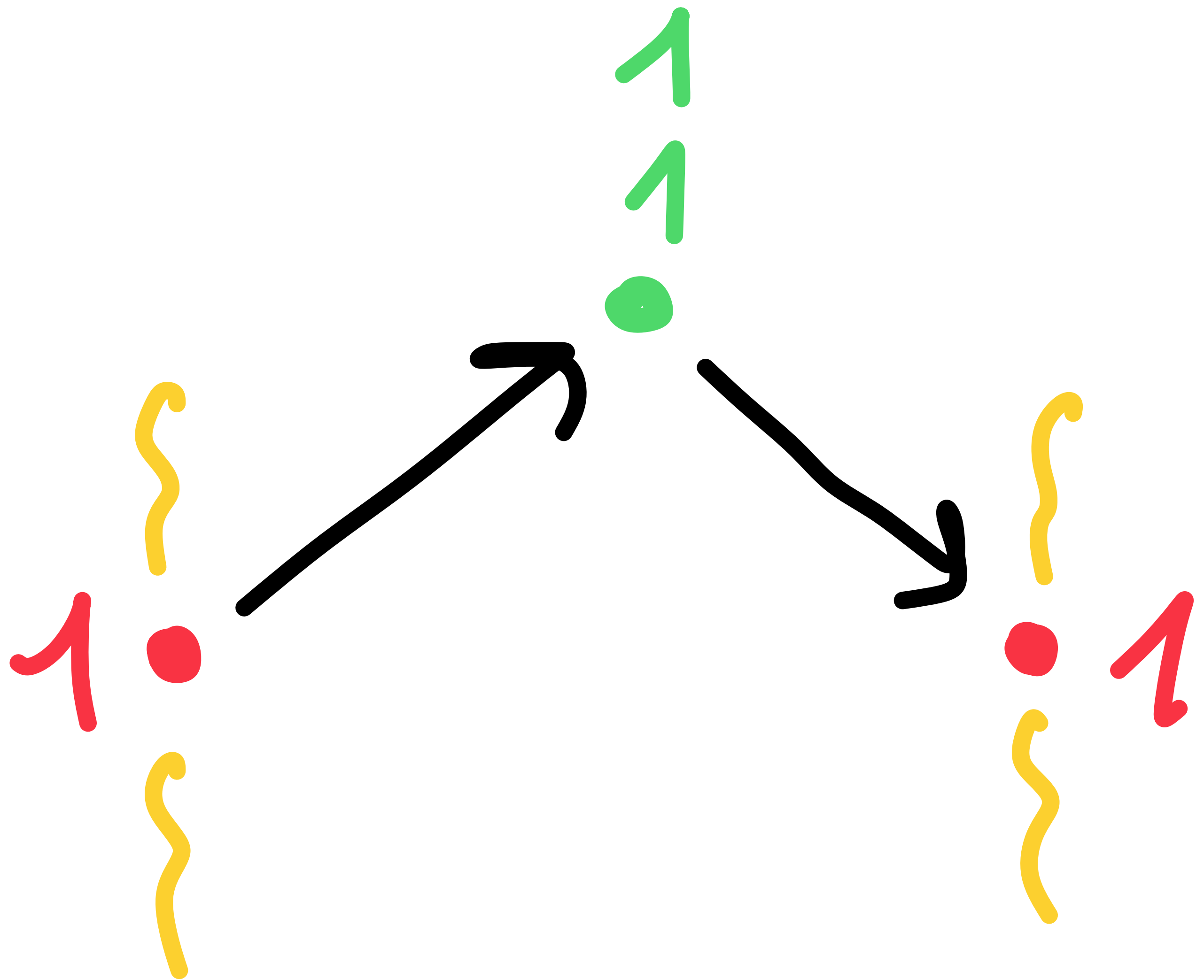
vertices $\longleftrightarrow$ indecomposable modules

arrows $\longleftrightarrow$ IRREDUCIBLE maps

$$A = K[x]/(x^2)$$

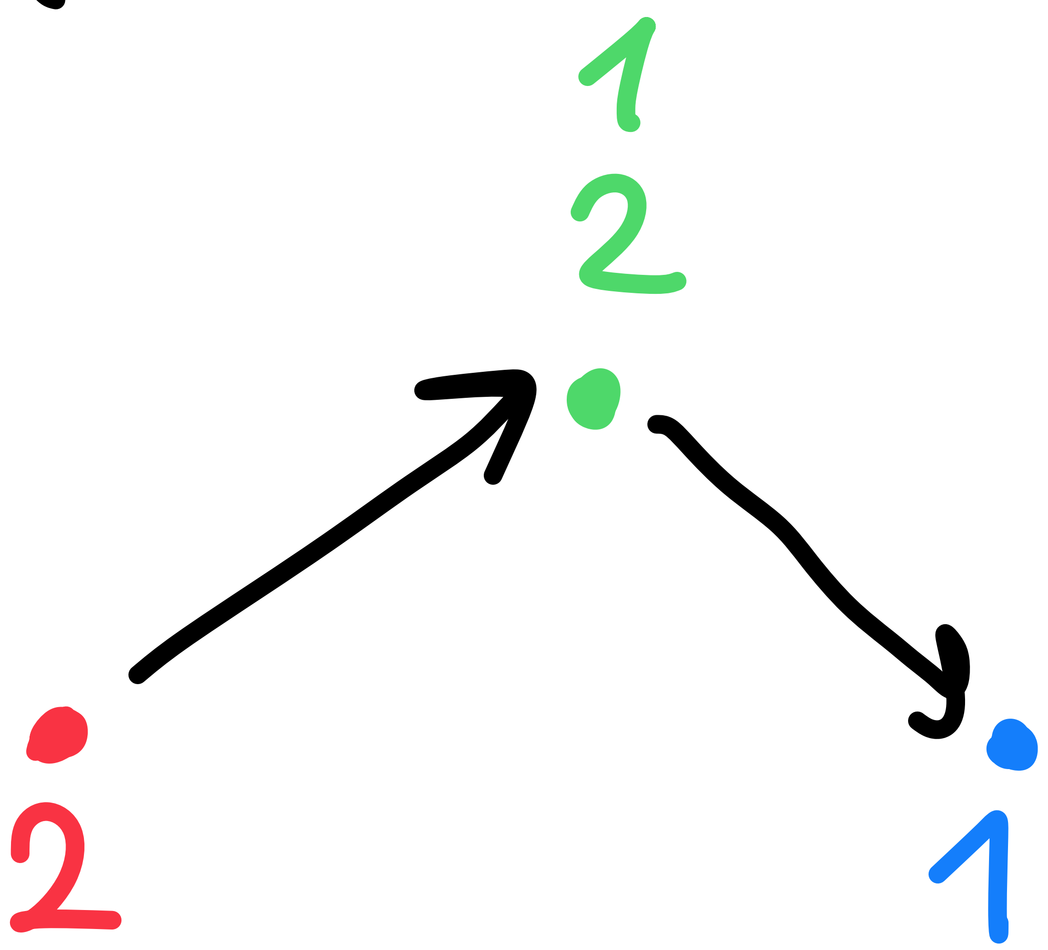
$$\begin{tikzcd} 1 \arrow[loop right, "a"] \end{tikzcd}$$

$$a^2 = 0$$



$$A = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$$

$$\begin{tikzcd} 1 \arrow[r] & 2 \end{tikzcd}$$



- Possible TITLE of this talk:

"Tilting theory for everyone"

- Tilting OBJECTS in this talk:

"Classical" tilting MODULES and

τ-tilting MODULES

A few words on Tilting modules:

Parents: S. BRENNER and M.C.R. BUTLER

Birthdate: ~ 1980

"20 years of Tilting Theory" = title of a Workshop near Munich in 2002

"The handbook of Tilting Theory" = title of the Proceedings of the Workshop

Almost always in this talk:

rings = finite dimensional algebras / K

modules = left modules of finite dim. / K

Many useful modules M are BASIC (or MULTIPLICITY FREE): $M = \bigoplus_{i=1}^{n} M_i$

where $M_1, \dots, M_n$ are { indecomposable, pairwise non $\cong$ }

Over a finite dim. algebra A with n simple modules a basic module T is a <u>basic tilting module</u> if

- $\operatorname{proj\,dim} T \leq 1$
- $\operatorname{Ext}^1_A(T, T) = 0$
- $\boxed{T = T_1 \oplus \dots \oplus T_n \text{ with } T_1, \dots, T_n \text{ indecomp.}}$

D. HAPPEL proved that

we may replace • by •

• There is a short exact sequence

$0 \longrightarrow {}_A A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ where

T' and T' are $\oplus$ of summands of T

C.M. RINGEL proved that

T tilting module, S module s.t.

- $\operatorname{proj\,dim} T \oplus S \leq 1$
- $\operatorname{Ext}^1_A(T \oplus S, T \oplus S) = 0$

$\Longrightarrow$ $S = \bigoplus$ of summands of T

V partial tilting module :

- proj dim $V \leq 1$
- $\mathrm{Ext}^1_A(V,V) = 0$

$\therefore$ projective module $\Longrightarrow$ partial tilting module

WHY Only NON COMMUTATIVE rings R admit non obvious partial tilting modules :

COLPI - MENINI proved that

$$\left.\begin{array}{l} {}_R M \text{ finitely presented} \\ R \text{ commutative} \\ \operatorname{proj\,dim} M \leq 1 \\ \operatorname{Ext}^1_R(M,M) = 0 \end{array}\right\} \Longrightarrow M \text{ projective}$$

EXAMPLE 1 (a finitely generated NON projective tilting module T)

A algebra given by $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$ $\quad A_2$

$T = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1$ minimal injective cogenerator

$$A \simeq \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}, \quad T \simeq A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Big/ A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

List of the basic tilting modules over A:

- $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$ <u>projective</u> generator
- $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$ <u>injective</u> cogenerator

$$\mathrm{Ext}^1_A(1,2) \neq 0$$

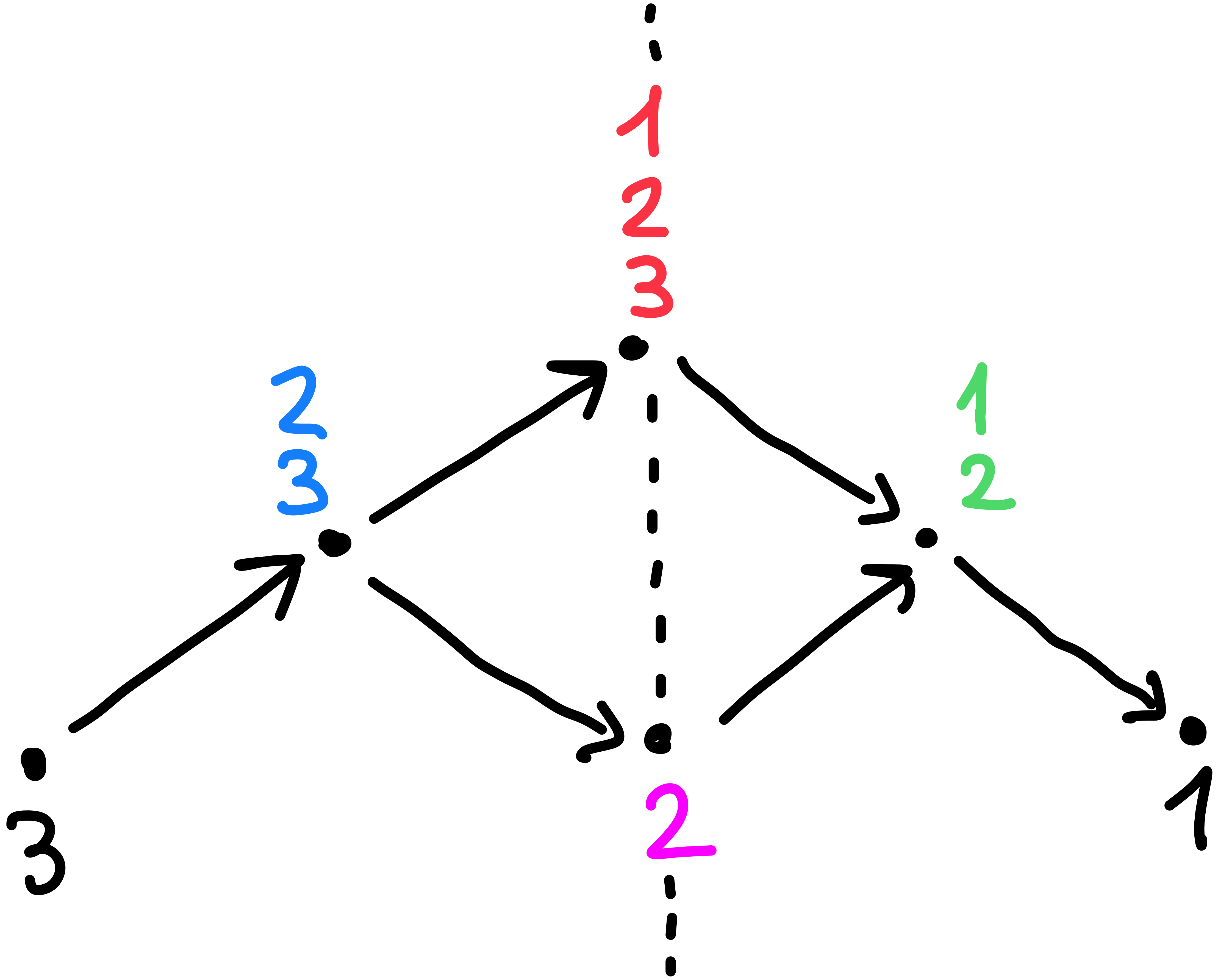
Auslander-Reiten quiver

$$\begin{tikzcd}[column sep=small] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\ 2 \arrow[ur] & & 1 \end{tikzcd}$$

Enough to replace $\bullet \longrightarrow \bullet$ by $\bullet \longrightarrow \bullet \longrightarrow \bullet$ to obtain an algebra with tilting modules admitting non projective and non injective indecomposable summands.

EXAMPLE 2 (2 tilting modules U, W with an indecomposable summand which is neither projective nor injective)

$$U = \underbrace{\boxed{\begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}2\\3\end{matrix}}}_{\text{projective}} \oplus 2 \quad , \quad W = \underbrace{\boxed{\begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}1\\2\end{matrix}}}_{\text{injective}} \oplus 2$$



R given by $\underbrace{\overset{\bullet}{1} \longrightarrow \overset{\bullet}{2} \longrightarrow \overset{\bullet}{3}}_{A_3}$

$$\mathrm{Ext}^1_A\left(\begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\right) \neq 0$$

$$R \simeq \begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{bmatrix}$$

List of the basic tilting modules over A:

$${}_A A = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \quad , \qquad I = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$$

$$U = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \quad , \qquad W = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$$

$$V = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1 \oplus 3$$

Starting point of our investigation:

In Examples 1 and 2, if X and Y are two basic tilting modules of the form $X = \bigoplus_{i=1}^{n} X_i$, $Y = \bigoplus_{j=1}^{n} Y_j$, then we OFTEN have

$$\boxed{\mathrm{Ext}_A^1(X_i, Y_j) \neq 0} \quad \text{or} \quad \boxed{\mathrm{Ext}_A^1(Y_j, X_i) \neq 0}.$$

NATURAL QUESTION

What is the <u>relationship</u> between the INDECOMPOSABLE summands of 2 basic tilting modules?

Lemma 1 (on tilting modules)

$\boxed{X = \bigoplus_{i=1}^{n} X_i}$, $\boxed{Y = \bigoplus_{i=1}^{n} Y_i}$ basic tilting modules

with n indecomposable summands

$$F(i) = \{ j \,/\, \boxed{X_i \simeq Y_j} \text{ or } \boxed{\mathrm{Ext}^1_A(X_i, Y_j) \oplus \mathrm{Ext}^1_A(Y_j, X_i) \neq 0} \}$$

$\Longrightarrow$

- $F(i) \neq \emptyset$ for any $i = 1, \dots, n$
- $|F(i_1) \cup \dots F(i_m)| \geq m$ if $i_1 < \dots < i_m$ and $m = 2, \dots, n$

Lemma 2 (without tilting modules)

$n \geq 2$, $F(1), \ldots, F(n)$ subsets of $\{1, \ldots, n\}$ s.t.

- $F(i) \neq \emptyset$ for any i
- $|F(i_1) \cup \ldots \cup F(i_m)| \geq m$ if $i_1 < \ldots < i_m$, $m \geq 2$

$\Longrightarrow$ For any $m = 1, \ldots, n-1$ $\exists$ an injective map $s : \{1, \ldots, m+1\} \to \{1, \ldots, n\}$ s.t. $\boxed{s(i) \in F(i)}$ for any i

WARNING The proof of Lemma 2 is **NOT** by induction.

Example If $m=3$, $n=4$, $F(1)=\{1,2\}$, $F(2)=\{2,3\}$, $F(3)=\{3,4\}$, $F(4)=\{1\}$, then the <u>injective</u> map $g:\{1,2,3\}\to\{1,2,3,4\}$ s.t. $g(i)=i$ for $i=1,2,3$ satisfies $\boxed{g(i)\in F(i)}$, but the unique map s satisfying Lemma 2 is $s=(1\,2\,3\,4)$.

Theorem A algebra with n simple modules, $X = \bigoplus_{i=1}^{n} X_i$, $Y = \bigoplus_{i=1}^{n} Y_i$ basic tilting modules

$\Longrightarrow$ $\exists$ a permutation $s \in S_n$ such that if $i = 1, \dots, n$ then either $\boxed{Y_{s(i)} \simeq X_i}$ or

$$\boxed{\mathrm{Ext}^1_A(X_i, Y_{s(i)}) \oplus \mathrm{Ext}^1_A(Y_{s(i)}, X_i) \neq 0}.$$

Example 3

A given by $1 \longrightarrow 3 \longleftarrow 2$ $\quad A_3$

isomorphic to $\begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ K & K & K \end{pmatrix}$

$$X = \boxed{\begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}} \oplus \begin{smallmatrix}1\,2\\3\end{smallmatrix} \qquad Y = \boxed{\begin{smallmatrix}1\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}} \oplus 3$$

$$\mathrm{Ext}^1_A\left(\begin{smallmatrix}1\,2\\3\end{smallmatrix}, 3\right) \neq 0$$

$$\begin{tikzcd}
 & \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[dr] & & 2 \\
3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\,2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & & 1
\end{tikzcd}$$

Example 4

A given by $\boxed{1 \longrightarrow 3 \longleftarrow 2}$ A_3

$$X = \boxed{\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix}} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$$

$$Z = \boxed{\begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix}} \oplus 1$$

$$\mathrm{Ext}^1_A\left(1, \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \neq 0$$

$$\begin{tikzcd}
 & \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \arrow[dr] & & 2 \\
3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix} 1\,2 \\ 3 \end{smallmatrix} \arrow[ur] \arrow[dr] & \\
 & \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \arrow[ur] & & 1
\end{tikzcd}$$

Example 5

A given by

$$\begin{tikzcd} & & 5 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 6 & 4 \arrow[l] & 3 \arrow[l] \end{tikzcd} \qquad E_6$$

$$X = \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} \oplus \begin{matrix} 2\ 5 \\ 6 \end{matrix} \oplus \begin{matrix} 5\ 4 \\ 6 \end{matrix} \oplus \begin{matrix} 1 \quad 3 \\ 2\ 5\ 4 \\ 6 \end{matrix} \oplus \begin{matrix} 1 \\ 2\ 5 \\ 6 \end{matrix} \oplus \begin{matrix} 3 \\ 5\ 4 \\ 6 \end{matrix}$$

Happel-Ringel tilting module

$\dim X = 23$

$\dim Y = 13$

$$Y = \begin{matrix} 1 \\ 2 \\ 6 \end{matrix} \oplus \begin{matrix} 2 \\ 6 \end{matrix} \oplus \begin{matrix} 3 \\ 4 \\ 6 \end{matrix} \oplus \begin{matrix} 4 \\ 6 \end{matrix} \oplus \boxed{\begin{matrix} 5 \\ 6 \end{matrix}} \oplus 6 = {}_A A$$

$\boxed{Y_5 = \begin{matrix} 5 \\ 6 \end{matrix} = X_1}$ and $\exists$ non split exact sequences

$$0 \to Y_1 = \begin{smallmatrix} 1 \\ 2 \\ 6 \end{smallmatrix} \to \begin{smallmatrix} 1 & & & & & & 3 \\ & 2 & & 5 & & 4 & \\ & & 6 & & 6 & & \end{smallmatrix} \to \begin{smallmatrix} & & 3 \\ 5 & & 4 \\ & 6 & \end{smallmatrix} = X_6 \to 0$$

$$0 \to Y_6 = 6 \to \boxed{\begin{smallmatrix} 1 \\ 2 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 6 \end{smallmatrix}} \to \begin{smallmatrix} 1 & & & & & & 3 \\ & 2 & & 5 & & 4 & \\ & & 6 & & 6 & & \end{smallmatrix} = X_4 \to 0$$

$$0 \to Y_4 = \begin{smallmatrix} 4 \\ 6 \end{smallmatrix} \to \boxed{\begin{smallmatrix} 2 & & 5 & & 4 \\ & 6 & & 6 & \end{smallmatrix}} \to \begin{smallmatrix} 2 & & 5 \\ & 6 & \end{smallmatrix} = X_2 \to 0$$

$$0 \to Y_2 = \begin{smallmatrix} 2 \\ 6 \end{smallmatrix} \to \boxed{\begin{smallmatrix} 2 & & 5 & & 4 \\ & 6 & & 6 & \end{smallmatrix}} \to \begin{smallmatrix} 5 & & 4 \\ & 6 & \end{smallmatrix} = X_3 \to 0$$

$$0 \to Y_3 = \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \to \begin{smallmatrix} 1 & & & & & & 3 \\ & 2 & & 5 & & 4 & \\ & & 6 & & 6 & & \end{smallmatrix} \to \begin{smallmatrix} 1 & & & \\ & 2 & & 5 \\ & & 6 & \end{smallmatrix} = X_5 \to 0$$

A permutation s as in the Theorem is

$$s = (1\ 5\ 3\ 2\ 4\ 6)$$

The middle term of the 5 exact sequences

$0 \to Y_{s(i)} \to \square \to X_i \to 0$ with $i>1$ can be

- $\begin{array}{ccc} 2 & 5 & 4 \\ & 6\ 6 & \end{array}$ indecomp. but not a summand of $X \oplus Y$
- $\begin{array}{c} 1 \\ 2 \\ 6 \end{array} \oplus \begin{array}{c} 5 \\ 6 \end{array} \oplus \begin{array}{c} 3 \\ 4 \\ 6 \end{array}$ $\oplus$ of 3 indec. summands of Y

The previous definition of TILTING MODULES was generalized in many directions:

- finite dim. algebras $\rightsquigarrow$ rings
- $1 \rightsquigarrow n \in \mathbb{N}$
- modules $\rightsquigarrow$ complexes
  $\rightsquigarrow$ more abstract objects

By dealing with a f.dim. algebra A

$\tau : \mathrm{mod}\, A \to \mathrm{mod}\, A$ is a map such that

- $\tau(M) = 0 \iff M$ is projective
- $\tau(\oplus M_i) = \oplus\, \tau(M_i)$
- τ induces a bijection

$\{\text{indec. non projective}\} \longrightarrow \{\text{indec. non injective}\}$

- M indec. non proj. $\implies \exists$ $0 \to \tau(M) \to L \to M \to 0$

non split exact sequence

2 definitions

X f.dim. module over a f.dim algebra A
with n simple modules

- X τ-rigid : $\mathrm{Hom}_A(X, \tau(X)) = 0$
- X τ-tilting : X basic, τ-rigid and
  $X = X_1 \oplus \dots \oplus X_n$ with
  $X_1, \dots, X_n$ indecomp.

Known facts:

- tilting module $\Longrightarrow$ τ-tilting module
- τ-tilting module $\not\Longrightarrow$ tilting module
- $\boxed{\text{$\tau$-tilting + faithful module}}$ $\Longrightarrow$ tilting module

Proposition In the hypotheses of the Theorem we **CANNOT** replace

X and Y basic tilting

by

X and Y basic τ-tilting

EVEN if one of the modules X and Y is tilting.

Example 6

A given by $1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ with $ba = 0$, $ab = 0$

$X = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ tilting

$Y = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1$ τ-tilting

$$1 \to \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to 2 \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to 1$$

$$\operatorname{Ext}^1_A\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 1\right) \oplus \operatorname{Ext}^1_A\left(1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = 0$$

Remark $\boxed{\begin{array}{l} X \text{ tilting},\ Y\ \tau\text{-tilting} \\ \exists\, s \in S_n \text{ as in the Theorem} \end{array}}$ $\not\Longrightarrow$

Y is tilting

Example 7 A given by
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[loop right] \end{tikzcd}$$
with $xy = 0$ for all x, y

$$X = \boxed{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}} \oplus \begin{smallmatrix}3\\3\end{smallmatrix}, \quad Y = \boxed{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}} \oplus 2$$

X tilting

Y τ-tilting

Auslander-Reiten quiver of A :

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 3 & & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] & & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ddr] & \\
3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\,3\\3\end{smallmatrix} \arrow[uur] \arrow[dr] & & & \\
 & \begin{smallmatrix}3\\3\end{smallmatrix} \arrow[ur] & & 2 \arrow[uur] & & 1
\end{tikzcd}$$

$$\therefore \; \mathrm{Ext}^1_A\left(2, \begin{smallmatrix}3\\3\end{smallmatrix}\right) \neq 0 \implies \exists \; s \in S_n \, \cdots\cdots$$

Thank you for your attention !

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For a visual presentation of Happel-Ringel tilting module, go to Google and write

Tilting Theory: a gift of Representation Theory to Mathematics